BILINEAR SPHERICAL MAXIMAL FUNCTION

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ABSTRACT. We obtain boundedness for the bilinear spherical maximal function in a range of exponents that includes the Banach triangle and a range of L^p with $p < 1$. We also obtain counterexamples that are asymptotically optimal with our positive results on certain indices as the dimension tends to infinity.

1. INTRODUCTION

Let σ be surface measure on the unit sphere. The spherical maximal function

$$\mathscr{M}(f)(x) = \sup_{t>0} \Big| \int_{|y|=1} f(x-ty)\,d\sigma(y) \Big|, \tag{1}$$

was first studied by Stein [23] who provided a counterexample showing that it is unbounded on $L^p(\mathbb{R}^n)$ for $p \le \frac{n}{n-1}$ and obtained the a priori inequality $\|\mathscr{M}(f)\|_{L^p(\mathbb{R}^n)} \le C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$ when $n \ge 3$, $p \in (\frac{n}{n-1}, \infty)$ for smooth functions f; see also the account in [24, Chapter XI]. The extension of this result to the case $n = 2$ was established about a decade later by Bourgain [1].

In addition to Stein and Bourgain, other authors have studied the spherical maximal function; for instance see [6], [3], [21], [19], and [22]. Among the techniques used in these works, we highlight that of Rubio de Francia [21], in which the L^p boundedness of (1) is reduced to certain L^2 estimates obtained by Plancherel's theorem. Extensions of the spherical maximal function to different settings have also been established by several authors: for instance see [5], [2] [15], [9] and [18].

In this work we study the bi(sub)linear spherical maximal function defined in (2), which was introduced and first studied by Geba, Greenleaf, Iosevich, Palsson, and Sawyer [10]; in this reference the authors obtained an $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ bound for (2). A multilinear (non-maximal) version of this operator when all input functions lie in the same space $L^p(\mathbb{R})$

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was previously studied by Oberlin [20]. Although most of our positive results focus on the case $n \geq 2$, a related recent paper of Greenleaf, Iosevich, Krause, and Liu [16] addresses the study of a related bilinear circular average when $n = 1$.

In the bilinear setting the role of the crucial $L^2 \to L^2$ estimate is played by an $L^2 \times L^2 \to L^1$, and obviously Plancherel's identity cannot be used on L^1. We overcome the lack of orthogonality on L^1 via a wavelet technique introduced by three of the authors in [13] in the study of certain bilinear operators; on this approach see [14], [17]. Our object of study here is the bi(sub)linear spherical maximal function

$$\mathcal{M}(f,g)(x) = \sup_{t>0} \Big| \int_{\mathbb{S}^{2n-1}} f(x-ty)g(x-tz)\,d\sigma(y,z) \Big| \tag{2}$$

initially defined for Schwartz functions f, g on $\mathbb{R}^n$. Here σ is surface measure on the $(2n-1)$-dimensional sphere. We are concerned with bounds for $\mathcal{M}$ from a product of Lebesgue spaces $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to another Lebesgue space $L^p(\mathbb{R}^n)$, where $1/p = 1/p_1 + 1/p_2$. The main result of this article is the following:

Theorem 1. *Let $n \geq 8$ and let $\delta_n = (2n-15)/10$. Then the bilinear maximal operator $\mathcal{M}$, when restricted to Schwartz functions, is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ for all indices $(\frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p})$ in the open rhombus with vertices the points $\vec{P}_0 = (\frac{1}{\infty}, \frac{1}{\infty}, \frac{1}{\infty})$, $\vec{P}_1 = (1, \frac{1}{\infty}, 1)$, $\vec{P}_2 = (\frac{1}{\infty}, 1, 1)$ and $\vec{P}_3 = (\frac{1+2\delta_n}{2+2\delta_n}, \frac{1+2\delta_n}{2+2\delta_n}, \frac{1+2\delta_n}{1+\delta_n})$.*

In Section 6 we give counterexamples indicating that this result is optimal, in the sense that, the difference between the range of p's in the positive result and counterexample tends to 0 as the dimension increases to ∞.

Once Theorem 1 is known, it follows that $\mathcal{M}$ admits a bounded extension from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for indices in the open rhombus of Theorem 1 (for such indices we have $p_1, p_2 < \infty$). Indeed, given $\{f_j\}_j$ Schwartz functions converging to f in L^{p_1} and $\{g_k\}_k$ Schwartz functions converging to g in L^{p_2}, we have that

$$\|\mathcal{M}(f_j, g_j) - \mathcal{M}(f_{j'}, g_{j'})\|_{L^p} \leq \big\|\mathcal{M}(f_j - f_{j'}, g_j) + \mathcal{M}(f_j, g_j - g_{j'})\big\|_{L^p}.$$

It follows from this that the sequence $\{\mathcal{M}(f_j, g_j)\}_j$ is Cauchy in $L^p(\mathbb{R}^n)$ and hence it converges to a value which we also call $\mathcal{M}(f,g)$. This is the bounded extension of $\mathcal{M}$ from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. In order to pass to the maximal function defined on $L^{p_1} \times L^{p_2}$, it is also possible to used the technique desribed in [24, page 508].

Concerning dimensions smaller than 8, we have positive answers in the Banach range in next section.

Remark 1. The proof of Theorem 1 only uses the decay of $\widehat{d\sigma}$ and its derivative, so it could be extended to more general surfaces with non-vanishing curvature whose associated surface measure satisfies similar decay estimates. For the sake of simplicity, however, in this work we focus attention only on the sphere.

2. THE BANACH RANGE IN DIMENSIONS $n \geq 2$

Proposition 2. *Let $n \geq 2$. Then $\mathcal{M}$ maps $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $1 < p_1, p_2 \leq \infty$, and $1 < p \leq \infty$.*

Proof. We show that $\mathcal{M}$ is bounded on the intervals $[\vec{P}_0, \vec{P}_1)$ and $[\vec{P}_0, \vec{P}_2)$, where $\vec{P}_1$ and $\vec{P}_2$ are as in Theorem 1. Then the claimed assertion follows by interpolation. If one function, for instance the second one g, lies in L^∞, matters reduce to the $L^p(\mathbb{R}^n)$ boundedness of the maximal operator

$$\mathcal{M}^0(f)(x) = \sup_{t>0} \int_{\mathbb{S}^{2n-1}} |f(x-ty)| d\sigma(y,z),$$

since $\mathcal{M}(f,g)(x) \leq \|g\|_{L^\infty} \mathcal{M}^0(f)(x)$. This expression inside the supremum is a Fourier multiplier operator of the form

$$\int_{\mathbb{R}^{2n}} \widehat{|f|}(\xi)\delta_0(\eta)\widehat{d\sigma}(t\xi, t\eta)e^{2\pi i x\cdot(\xi+\eta)} d\xi d\eta = \int_{\mathbb{R}^n} \widehat{|f|}(\xi)\widehat{d\sigma}(t\xi, 0)e^{2\pi i x\cdot\xi} d\xi$$

where δ_0 is the Dirac mass and

$$\widehat{d\sigma}(t(\xi,0)) = 2\pi \frac{J_{n-1}(2\pi t|(\xi,0)|)}{|t(\xi,0)|^{n-1}}.$$

The multiplier $\widehat{d\sigma}(\xi,0)$ is smooth everywhere and decays like $|\xi|^{-(n-\frac{1}{2})}$ as $|\xi| \to \infty$ and its gradient has a similar decay.

The following result is in [21, Theorem B] (see also [8]):

Theorem A. *Let $m(\xi)$ be a $\mathcal{C}^{[n/2]+1}(\mathbb{R}^n)$ function that satisfies $|\partial^\gamma m(\xi)| \leq (1+|\xi|)^{-a}$ for all $|\gamma| \leq [n/2]+1$ with $a \geq (n+1)/2$. Then the maximal operator*

$$f \mapsto \sup_{t>0} \left| \left(\widehat{f}(\xi)m(t\xi)\right)^\vee \right|$$

maps $L^p(\mathbb{R}^n)$ to itself for $1 < p < \infty$.

In order to have $n - \frac{1}{2} \geq \frac{n+1}{2}$ we must assume that $n \geq 2$. It follows from Theorem A that $\mathcal{M}^0$ is bounded on L^p when $1 < p \leq \infty$ and $n \geq 2$. This completes the proof of Proposition 2.

□

Remark 2. For $n \geq 5$, using the result of Cho [4] (which provides an extension of Rubio de Francia's theorem [21] in the endpoint $p = 1$) one may obtain that $\mathcal{M}$ maps continuously $H^1 \times L^\infty$ into L^1. Here H^1 is the Hardy space.

3. THE POINT $(2,2,1)$

Next we turn to the main estimate of this article which concerns the point $L^2 \times L^2 \to L^1$, i.e., the estimate $\|\mathcal{M}(f,g)\|_{L^1} \le \|f\|_{L^2}\|g\|_{L^2}$.

Proposition 3. *If ψ is in $C_0^\infty(\mathbb{R}^{2n})$, then the maximal function*

$$M(f,g)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\psi(t\xi,t\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta\right|$$

satisfies that for any $1<p_1,p_2<\infty$ and $1/p = 1/p_1 + 1/p_2$, there exists a constant C independent f and g such that

$$\|M(f,g)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}$$

The proof of Proposition 3 is standard and is omitted. Next, we decompose $\mathcal{M}$. We fix $\varphi_0 \in C_0^\infty(\mathbb{R}^{2n})$ such that $\chi_{B(0,1)} \le \varphi_0 \le \chi_{B(0,2)}$ and we let $\varphi(\xi,\eta) = \varphi_0((\xi,\eta)) - \varphi_0(2(\xi,\eta))$. For $j \ge 1$ define

$$m_j(\xi,\eta) = \widehat{d\sigma}(\xi,\eta)\varphi(2^{-j}(\xi,\eta))$$

and for $j=0$ define $m_0(\xi,\eta) = \widehat{d\sigma}(\xi,\eta)\varphi_0(\xi,\eta)$. Then we have

$$\widehat{d\sigma} = m = \sum_{j\ge 0} m_j$$

where $\widehat{d\sigma}(\xi,\eta) = 2\pi\frac{J_{n-1}(2\pi(\xi,\eta))}{|(\xi,\eta)|^{n-1}}$. Setting

$$\mathcal{M}_j(f,g)(x) = \sup_{t>0}\left|\int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)m_j(t\xi,t\eta)e^{2\pi i x\cdot(\xi+\eta)}d\xi d\eta\right|,$$

we have the pointwise estimate

$$\mathcal{M}(f,g)(x) \le \sum_{j\ge 0}\mathcal{M}_j(f,g)(x), \qquad x \in \mathbb{R}^n. \tag{3}$$

Proposition 4. *For $n \ge 8$, there exist positive constants C and $\delta_n = \frac{n}{5} - \frac{3}{2}$ such that for all $j \ge 1$ and all functions $f,g \in L^2(\mathbb{R}^n)$ we have*

$$\|\mathcal{M}_j(f,g)\|_{L^1} \le C\, j 2^{-\delta_n j}\|f\|_{L^2}\|g\|_{L^2}. \tag{4}$$

Proposition 4 will be proved in the next section. In the remaining of this section we state and prove a lemma needed for its proof.

Lemma 5. *Suppose that $\sigma_1(\xi,\eta)$ is defined on $\mathbb{R}^{2n}$ and for some $\delta>0$ it satisfies:*

(i) for any multiindex $|\alpha| \le M = 4n$, there exists a positive constant C_α independent of j such that $\|\partial^\alpha(\sigma_1(\xi,\eta))\|_{L^\infty} \le C_\alpha 2^{-j\delta}$,

(ii) supp $\sigma_1 \subset \{(\xi,\eta)\in\mathbb{R}^{2n} : |(\xi,\eta)|\sim 2^j, c_1 2^{-j}\le \frac{|\xi|}{|\eta|}\le c_2 2^j\}$.

Then $T(f,g)(x) := \int_0^\infty |T_{\sigma_t}(f,g)(x)|\frac{dt}{t}$ *is bounded from* $L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ *to* $L^1(\mathbb{R}^n)$ *with bound at most a multiple of* $j\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}$*, where* $\sigma_t(\xi,\eta) = \sigma_1(t\xi,t\eta)$.

Proof of Lemma 5. A crucial tool in the proof of Lemma 5 is the following:

Proposition B. *Let* $m\in L^2(\mathbb{R}^{2n})$ *and* $C_M>0$ *satisfy* $\|\partial^\alpha m\|_{L^\infty}\le C_M$ *for each multiindex* $|\alpha|\le M = 16n$*. Then the bilinear operator* T_m *associated with the multiplier* m *satisfies*

$$\|T_m\|_{L^2\times L^2\to L^1}\le C C_M^{1/5}\|m\|_{L^2}^{4/5}. \tag{5}$$

The proof of Proposition B (stated as Corollary 8 in [13]) requires a delicate wavelet technique and is implicitly contained in [13, Section 4]. For the sake of completeness, we include the proof in the appendix at the end of the paper.

Using Proposition B, setting $\widehat{f^j} = \widehat{f}\chi_{\{c_1\le|\xi|\le c_2 2^{j+1}\}}$, by the support of σ_1 we obtain that

$$\|T_{\sigma_1}(f,g)\|_{L^1}\le C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}\|f^j\|_{L^2}\|g^j\|_{L^2}.$$

Notice that $T_{\sigma_t}(f,g)(x) = t^{-2n}T_{\sigma_1}(f_t,g_t)(\frac{x}{t})$, where $\widehat{f_t}(\xi) = \widehat{f}(\xi/t)$. Then

$$\begin{aligned}\|T_{\sigma_t}(f,g)\|_{L^1} &\le C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}t^{-n}\|\widehat{f}(\xi/t)\chi_{E_{j,0}}\|_{L^2}\|\widehat{g}(\eta/t)\chi_{E_{j,0}}\|_{L^2}\\ &= C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}\|\widehat{f}\chi_{E_{j,t}}\|_{L^2}\|\widehat{g}\chi_{E_{j,t}}\|_{L^2},\end{aligned}$$

where $E_{j,t} = \{\xi\in\mathbb{R}^n : \frac{c_1}{t}\le|\xi|\le\frac{2^j c_2}{t}\}$.

As a result we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_0^\infty |T_{\sigma_t}(f,g)|\frac{dt}{t}dx\\ \le& C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}\int_0^\infty \|\widehat{f}\chi_{E_{j,t}}\|_{L^2}\|\widehat{g}\chi_{E_{j,t}}\|_{L^2}\frac{dt}{t}\\ \le& C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}\left(\int_0^\infty\int_{\mathbb{R}^n}|\widehat{f}\chi_{E_{j,t}}|^2 d\xi\frac{dt}{t}\right)^{\frac12}\left(\int_0^\infty\int_{\mathbb{R}^n}|\widehat{g}\chi_{E_{j,t}}|^2 d\xi\frac{dt}{t}\right)^{\frac12}.\end{aligned}$$

We control the last term as follows:

$$\int_0^\infty\int_{\mathbb{R}^n}|\widehat{f}\chi_{E_{j,t}}|^2 d\xi\frac{dt}{t}\le C\int_{\mathbb{R}^n}\int_{1/|\xi|}^{2^j/|\xi|}\frac{dt}{t}|\widehat{f}(\xi)|^2 d\xi\le Cj\|f\|_{L^2}^2$$

and thus we deduce

$$\|T(f,g)(x)\|_{L^1}\le C\|\sigma_1\|_{L^2}^{4/5}2^{-j\delta/5}j\|f\|_{L^2}\|g\|_{L^2}.$$

This completes the proof of Lemma 5. □

We note that $C_M^{1/5}$ captures the decay (if any) of the L^∞ norms of the derivatives of the multipliers. This is the situation we encounter in the next section.

4. PROOF OF PROPOSITION 4

Proof. Estimate (4) is automatically holds for finitely many terms in view of Proposition 3, so we fix a large j and define

$$T_{j,t}(f,g)(x) = \int_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)m_j(t\xi,t\eta)e^{2\pi ix\cdot(\xi+\eta)}d\xi d\eta.$$

Take a smooth function ρ on $\mathbb{R}$ such that $\chi_{[\varepsilon-1,1-\varepsilon]} \le \rho \le \chi_{[-1,1]}$. Define $m_j^1(\xi,\eta) = m_j(\xi,\eta)\rho(\frac{1}{j}(\log_2\frac{|\xi|}{|\eta|}))$, then we have a smooth decomposition of m_j with $m_j = m_j^1 + m_j^2$. On the support of m_j^1 we have $C^{-1}2^{-j}|\xi| \le |\eta| \le C2^j|\xi|$ and on the support of m_j^2 we have $2^{j(1-\varepsilon)}|\xi| \lesssim |\eta|$ or $2^{j(1-\varepsilon)}|\eta| \lesssim |\xi|$. We define

$$\mathcal{M}_j^i(f,g) = \sup_{t>0}|T_{j,t}^i(f,g)|, \quad i \in \{1,2\},$$

where $T_{j,t}^1$ and $T_{j,t}^2$ correspond to multipliers $m_j^1(t(\xi,\eta))$ and $m_j^2(t(\xi,\eta))$ respectively, such that $T_{j,t} = T_{j,t}^1 + T_{j,t}^2$. Then for f,g Schwartz functions we have

$$\begin{aligned}\mathcal{M}_j^1(f,g)(x) &= \sup_{t>0}|T_{j,t}^1(f,g)(x)| \\ &= \sup_{t>0}\left|\int_0^t s\frac{dT_{j,s}^1(f,g)}{ds}\frac{ds}{s}\right| \\ &\le \int_0^\infty |\widetilde{T}_{j,s}^1(f,g)(x)|\frac{ds}{s},\end{aligned}$$

where $\widetilde{T}_{j,s}^1$ has bilinear multiplier $\widetilde{m}_j^1(s\xi,s\eta) = (s\xi,s\eta)\cdot(\nabla m_j^1)(s\xi,s\eta)$, a diagonal multiplier with nice decay, which can be used to establish the boundedness of the diagonal part with the aid of Lemma 5.

Recall that

$$m_j^1(\xi,\eta) = \varphi(2^{-j}(\xi,\eta))2\pi\frac{J_{n-1}(2\pi(\xi,\eta))}{|(\xi,\eta)|^{n-1}}\rho(\tfrac{1}{j}(\log_2\tfrac{|\xi|}{|\eta|}))$$

for $j \ge 1$ and a calculation shows that $|\partial_1(m_j^1)|$ is controlled by the sum of three terms bounded by $C2^{-j(2n-1)/2}$, $C2^{-j(2n+1)/2}$ and $C\frac{1}{j}2^{-j(2n-1)/2}$ respectively. Indeed, when the derivative falls on ϕ, we can bound it by

$C2^{-j}2^{-j(n-1/2)} = C2^{-j(n+1/2)}$. If the derivative falls on the second part, using properties of Bessel functions (see, e.g., [11, Appendix B.2]), we obtain the bound $C\frac{J_n(2\pi(\xi,\eta))}{|(\xi,\eta)|^n}|\xi_1| \leq C2^{-j(n-1/2)}$. For the last case, we can bound it by $C2^{-j(n-1/2)}j^{-1}\frac{1}{|\xi|}\frac{\xi_1}{|\xi|} \leq C2^{-j(n-1/2)}j^{-1}2^{-\varepsilon j}$. As a consequence we have $|\partial_1(m_j^1)| \leq C2^{-j(2n-1)/2}$. Then we can show that $|\partial_1(\widetilde{m}_j^1)| \leq C2^{-j(2n-3)/2}$ and similar arguments give that for any multiindex α we have $|\partial^\alpha \widetilde{m}_j^1| \leq C2^{-j(2n-3)/2}$. Moreover, from this we can show that

$$\|\widetilde{m}_j^1\|_2 \leq C\bigg(\int_{|(\xi,\eta)|\sim 2^j} |2^{-j(n-\frac{3}{2})}|^2 d\xi d\eta\bigg)^{\frac{1}{2}} \leq C2^{-j(n-\frac{3}{2})}2^{jn} \leq C2^{\frac{3}{2}j}.$$

Applying Lemma 5 to the function $\widetilde{m}_j^1(\xi,\eta) = (\xi,\eta)\cdot(\nabla m_j^1)(\xi,\eta)$ which satisfies the hypotheses with $\delta = (2n-3)/2$, we obtain

$$\|\mathcal{M}_j^1(f,g)\|_{L^1} \leq Cj\|\widetilde{m}_j^1\|_{L^2}^{\frac{4}{5}} 2^{-j\frac{\delta}{5}}\|f\|_{L^2}\|g\|_{L^2} = Cj2^{j(\frac{3}{2}-\frac{n}{5})}\|f\|_{L^2}\|g\|_{L^2}. \tag{6}$$

It remains to obtain an analogous estimate for $\mathcal{M}_j^2$.

For the off-diagonal part m_j^2 we use a different decomposition involving g-functions. For $f,g \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\begin{aligned}
\mathcal{M}_j^2(f,g)(x) &= \big(\sup_{t>0}|T_{j,t}^2(f,g)(x)|^2\big)^{\frac{1}{2}} \\
&= \bigg(\sup_{t>0}\bigg|2\int_0^t T_{j,s}^2(f,g)(x)\, s\frac{dT_{j,s}^2(f,g)(x)}{ds}\frac{ds}{s}\bigg|\bigg)^{\frac{1}{2}} \\
&\leq \sqrt{2}\Big\{\Big(\int_0^\infty |T_{j,s}^2(f,g)|^2\frac{ds}{s}\Big)^{\frac{1}{2}}\Big(\int_0^\infty |\widetilde{T}_{j,s}^2(f,g)|^2\frac{ds}{s}\Big)^{\frac{1}{2}}\Big\}^{\frac{1}{2}} \\
&= \sqrt{2}\big(G_j(f,g)(x)\widetilde{G}_j(f,g)\big)^{\frac{1}{2}}. \qquad (7)
\end{aligned}$$

Here $\widetilde{T}_{j,s}^2(f,g)$ has symbol $\widetilde{m}_j^2(s\xi,s\eta) = (s\xi,s\eta)\cdot(\nabla m_j^2)(s\xi,s\eta)$ and

$$\begin{aligned}
G_j(f,g)(x) &= \Big(\int_0^\infty |T_{j,s}^2(f,g)|^2\frac{ds}{s}\Big)^{\frac{1}{2}} \\
\widetilde{G}_j(f,g)(x) &= \Big(\int_0^\infty |\widetilde{T}_{j,s}^2(f,g)|^2\frac{ds}{s}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Lemma 6. *If a $\sigma_1(\xi,\eta)$ on $\mathbb{R}^{2n}$ satisfies*

(i) for any multiindex $|\alpha| \leq M = 4n$, there exists a positive constant C_α independent of j such that $\|\partial^\alpha(\sigma_1(\xi,\eta))\|_{L^\infty} \leq C_\alpha 2^{-j\delta}$,

(ii) supp $\sigma_1 \subset \{(\xi,\eta) \in \mathbb{R}^{2n} : |(\xi,\eta)| \sim 2^j, |\xi| \geq 2^{j(1-\varepsilon)}|\eta|$, or $|\eta| \geq 2^{j(1-\varepsilon)}|\xi|\}$,

then $T(f,g)(x) := (\int_0^\infty |T_{\sigma_t}(f,g)(x)|^2 \frac{dt}{t})^{1/2}$ *is bounded from* $L^2 \times L^2$ *to* L^1 *with bound at most a multiple of* $2^{-j(\delta-\varepsilon)}$*, where* $\sigma_t(\xi,\eta) = \sigma_1(t\xi, t\eta)$.

Proof. Recall that supp $m_j^2 \subset \{(\xi,\eta) : 2^{j(1-\varepsilon)}|\xi| \lesssim |\eta| \text{ or } 2^{j(1-\varepsilon)}|\eta| \lesssim |\xi|\}$. We consider only the part $\{|\xi| \geq 2^{j(1-\varepsilon)}|\eta|\}$ because the other part is similar. By [13, Section 5] we have

$$|T_{\sigma_1}(f,g)(x)| \leq C2^{\varepsilon j}2^{-j\delta}M(g)(x)|T_m(f)(x)|,$$

where M is the Hardy-Littlewood maximal function and T_m is a linear operator that satisfies $\|T_m(f)\|_{L^2} \leq C\|\widehat{f}\chi_{\{|\xi|\sim 2^j\}}\|_{L^2}$. Then

$$|T_{\sigma_t}(f,g)(x)| \leq 2^{-j(\delta-\varepsilon)}t^{-n}M(g)(x)T_m(f_t)(x/t),$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\int_0^\infty |T_{\sigma_t}(f,g)(x)|^2\frac{dt}{t}\right)^{\frac{1}{2}}dx \\
&\qquad\leq C2^{-j(\delta-\varepsilon)}\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-2n}M(g)(x)^2|T_m(f_t)(x/t)|^2\frac{dt}{t}\right)^{\frac{1}{2}}dx \\
&\qquad\leq C2^{-j(\delta-\varepsilon)}\|M(g)\|_{L^2}\left(\int_{\mathbb{R}^n}\int_0^\infty \left|t^{-n}T_m(f_t)(x/t)\right|^2\frac{dt}{t}dx\right)^{\frac{1}{2}} \\
&\qquad\leq C2^{-j(\delta-\varepsilon)}\|g\|_{L^2}\Big(\int_{\mathbb{R}^n}|\widehat{f}(\xi)|^2\int_{2^{j-1}/|\xi|}^{2^{j+1}/|\xi|}\frac{dt}{t}d\xi\Big)^{\frac{1}{2}} \\
&\qquad\leq C2^{-j(\delta-\varepsilon)}\|g\|_{L^2}\|f\|_{L^2}.
\end{aligned}$$

This completes the proof of Lemma 6. □

We now return to the proof of Proposition 4. Notice that both $m_j^2(\xi,\eta)$ and $\widetilde{m}_j^2(\xi,\eta)$ satisfy conditions of Lemma 6 with δ being either $(2n-1)/2$ or $(2n-3)/2$ respectively, so

$$\begin{aligned}
\|G_j(f,g)\|_{L^1} &\leq C2^{-j(2n-1)/2}\|f\|_{L^2}\|g\|_{L^2} \\
\|\widetilde{G}_j(f,g)\|_{L^1} &\leq C2^{-j(2n-3)/2}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

Using (7) we deduce

$$\|\mathcal{M}_j^2(f,g)\|_{L^1} \leq \|G_j(f,g)\|_{L^1}^{1/2}\|\widetilde{G}_j(f,g)\|_{L^1}^{1/2} \leq C2^{-j(n-1)}\|f\|_{L^2}\|g\|_{L^2}. \tag{8}$$

Combining (6) and (8) yields Proposition 4 with $\delta_n = \frac{n}{5} - \frac{3}{2}$. □

5. INTERPOLATION

By Proposition 3 (for term $j \le c_0$) and Proposition 4 (for $j \ge c_0$), for any $\delta_n' < \delta_n$, as a consequence of (3) we obtain

$$\|\mathcal{M}(f,g)\|_{L^1} \le \sum_{j=0}^{\infty} C_{\delta_0'} 2^{-\delta_n' j} \|f\|_{L^2}\|g\|_{L^2} \le C_{\delta_0'}\|f\|_{L^2}\|g\|_{L^2}.$$

This establishes the boundedness of $\mathcal{M}$ from $L^2 \times L^2$ to L^1 claimed in Theorem 1 (recall $n \ge 8$). It remains to obtain estimates for other values of p_1, p_2. This is achieved via bilinear interpolation.

Notice that when one index among p_1 and p_2 is equal to 1, we have that $\mathcal{M}_j$ maps $L^{p_1} \times L^{p_2}$ to $L^{p,\infty}$ with norm $\lesssim 2^j$. Indeed, this follows from the estimate

$$|\varphi_j^{\vee} * (d\sigma)(y,z)| \le C_N 2^j (1+|(y,z)|)^{-2N} \le C_N 2^j (1+|y|)^{-N}(1+|z|)^{-N}$$

which can be found, for instance, in [11, estimate (6.5.12)]. Thus we have

$$\mathcal{M}_j(f,g)(x) \le C 2^j M(f) M(g)$$

where M is the Hardy-Littlewood maximal function. We pick two points

$$\vec{Q}_1 = (1/1, 1/(1+\varepsilon), (2+\varepsilon)/(1+\varepsilon))$$
$$\vec{Q}_2 = (1/(1+\varepsilon), 1/1, (2+\varepsilon)/(1+\varepsilon))$$

and we also consider the point $\vec{Q}_0 = (1/2, 1/2, 1)$. We interpolate the known estimates for $\mathcal{M}_j$ at these three points. Letting ε go to 0, we obtain that for $p > \frac{2+2\delta_n}{1+2\delta_n}$ we have that $\mathcal{M}_j$ maps $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to $L^{p/2}(\mathbb{R}^n)$ with a geometrically decreasing bound in j. Recall that $\delta_n = (2n-15)/10 > 0$, so we need $n \ge 8$.

Thus summing over j gives boundedness for $\mathcal{M}$ from $L^p(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ to $L^{p/2}(\mathbb{R}^n)$ when $p > \frac{2+2\delta_n}{1+2\delta_n}$. By interpolation we obtain boundedness for $\mathcal{M}$ in the interior of a rhombus with vertices the points $(1/\infty, 1/\infty, 1/\infty)$, $(\frac{2n-3/2}{2n-1}, \frac{1}{\infty}, \frac{2n-3/2}{2n-1})$, $(\frac{1}{\infty}, \frac{2n-3/2}{2n-1}, \frac{2n-3/2}{2n-1})$ and $(\frac{1+2\delta_n}{2+2\delta_n}, \frac{1+2\delta_n}{2+2\delta_n}, \frac{2+4\delta_n}{2+2\delta_n})$. The proof of Theorem 1 is now complete.

We remark that is the largest region for which we presently know boundedness for $\mathcal{M}$ in dimensions $n \ge 8$.

6. COUNTEREXAMPLES

In this section we construct counterexamples indicating the unboundedness of the bilinear spherical maximal operator in a certain range. Our examples are inspired by Stein [23] but the situation is more complicated.

Proposition 7. *The bilinear spherical maximal operator $\mathcal{M}$ is unbounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1\le p_1,p_2\le\infty$, $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$, $n\ge 1$, and $p\le\frac{n}{2n-1}$. In particular, $\mathcal{M}$ is unbounded from $L^2(\mathbb{R})\times L^2(\mathbb{R})$ to $L^1(\mathbb{R})$ when $n=1$.*

Remark 3. We note that $\frac{1+\delta_n}{1+2\delta_n}-\frac{n}{2n-1}=\frac{1+\frac{n}{5}-\frac{3}{2}}{1+\frac{2n}{5}-3}-\frac{n}{2n-1}\approx\frac{1}{n}\to 0$ as $n\to\infty$. This means that the gap between the range of boundedness and unboundedness tends to 0 as the dimension increases to infinity.

Proof. We first consider the case $n=1$ where it is easy to demonstrate the main idea.

Define functions on $\mathbb{R}$ by setting $f(y)=|y|^{-1/p_1}(\log\frac{1}{|y|})^{-2/p_1}\chi_{|y|\le 1/2}$ and $g(y)=|y|^{-1/p_2}(\log\frac{1}{|y|})^{-2/p_2}\chi_{|y|\le 1/2}$. Then $f\in L^{p_1}(\mathbb{R})$, $g\in L^{p_2}(\mathbb{R})$ and we will estimate from below $M_{\sqrt{2}R}(f,g)(R)$ for large R, where

$$M_t(f,g)(x)=\int_{\mathbb{S}^1}|f(x-ty)g(x-tz)|d\sigma(y,z).$$

In view o the support properties of f and g we have $|y-\frac{1}{\sqrt{2}}|\le\frac{1}{2\sqrt{2}R}$, and $|z-\frac{1}{\sqrt{2}}|\le\frac{1}{2\sqrt{2}R}$. We also have that $y^2+z^2=1$ since $(y,z)\in\mathbb{S}^1$.

Therefore we rewrite $M_{\sqrt{2}R}(f,g)(R)$ as

$$\int_{\frac{\sqrt{2}}{2}-\frac{1}{2\sqrt{2}R}}^{\frac{\sqrt{2}}{2}+\frac{1}{2\sqrt{2}R}}|R(1-\sqrt{2}y)|^{-\frac{1}{p_1}}(-\log|R(1-\sqrt{2}y)|)^{-\frac{2}{p_1}} |R(1-\sqrt{2}z)|^{-\frac{1}{p_2}}(-\log|R(1-\sqrt{2}z)|)^{-\frac{2}{p_2}}\frac{dy}{\sqrt{1-y^2}}, \tag{9}$$

with $z=\sqrt{1-y^2}$.

Notice that $|R(1-\sqrt{2}z)|=R|\frac{1-2z^2}{1+\sqrt{2}z}|\le R|1-2y^2|\le 3R|1-\sqrt{2}y|$ since[1] $z\approx y\approx\sqrt{2}/2$. As a result, with the help of (10) [Lemma 8], the expression in (9) is greater than

$$\int_{\frac{\sqrt{2}}{2}-\frac{1}{100R}}^{\frac{\sqrt{2}}{2}+\frac{1}{100R}}R^{-\frac{1}{p}}|(1-\sqrt{2}y)|^{-\frac{1}{p}}(-\log|R(1-\sqrt{2}y)|)^{-\frac{2}{p}}dy$$
$$=2R^{-1}\int_0^{\frac{1}{100}}t^{-1/p}(\log\tfrac{1}{t})^{-2/p}dt=\begin{cases}C_pR^{-1} & \text{if } p\ge 1\\ \infty & \text{if } p<1.\end{cases}$$

[1]Here $a\approx b$ means that $|a-b|$ is very small.

Thus $\mathcal{M}(f,g) \notin L^p(\mathbb{R})$ for $p<1$ and also $\mathcal{M}(f,g)(x) \geq C/x$ for x large if $p=1$. It follows that $\mathcal{M}(f,g) \notin L^1(\mathbb{R})$ for $p=1$, hence the statement of the proposition holds.

We now consider the higher-dimensional case $n \geq 2$. We define $f(y) = |y|^{-n/p_1}(\log \frac{1}{|y|})^{-2/p_1}\chi_{|y|\leq 1/100}$ and $g(y) = |y|^{-n/p_2}(\log \frac{1}{|y|})^{-2/p_2}\chi_{|y|\leq 1/2}$. We have that f lies in $L^{p_1}(\mathbb{R}^n)$ and g lies in $L^{p_2}(\mathbb{R}^n)$. The mapping $(y,z) \mapsto (Ay,Az)$ with $A \in SO_n$ is an isometry on $\mathbb{S}^{2n-1}$, hence we have $M_t(f,g)(x) = M_t(f,g)(|x|e_1)$, where $e_1 = (1,0,\dots,0) \in \mathbb{R}^n$. Thus we may take $x = Re_1 \in \mathbb{R}^n$ with R large.

By the change of variables identity (11) [Lemma 9], we have

$$\begin{aligned} &M_{\sqrt{2}R}(f,g)(Re_1) \\ &\quad = \int_{\mathbb{S}^{2n-1}} f(Re_1 - \sqrt{2}Ry)g(Re_1 - \sqrt{2}Rz)\,d\sigma(y,z) \\ &\quad = \int_{B_n(\frac{1}{\sqrt{2}}e_1, \frac{1}{100R})} |\sqrt{R}y - Re_1|^{-\frac{n}{p_1}}(-\log|Re_1 - \sqrt{2}Ry|)^{-\frac{2}{p_1}} \\ &\qquad\quad \int_E |\sqrt{2}Rz - Re_1|^{-\frac{n}{p_2}}(-\log|Re_1 - \sqrt{2}Rz|)^{-\frac{2}{p_2}}\,d\sigma^r_{n-1}(z)\frac{dy}{\sqrt{1-|y|^2}}, \end{aligned}$$

where $B_n(a,r)$ is a ball in $\mathbb{R}^n$ centered at a with radius r, and E is the $(n-1)$-dimensional manifold $\mathbb{S}^{n-1}_{\sqrt{1-|y|^2}} \cap B_n(\frac{1}{\sqrt{2}}e_1, \frac{1}{2\sqrt{2}R})$ with $\mathbb{S}^{n-1}_r$ being the sphere in $\mathbb{R}^n$ with radius r and $d\sigma^r_{n-1}$ the measure on $\mathbb{S}^{n-1}_r$.

We next focus on the inner integral, namely

$$I = \int_E |\sqrt{2}Rz - Re_1|^{-\frac{n}{p_2}}(-\log|Re_1 - \sqrt{2}Rz|)^{-\frac{2}{p_2}}\,d\sigma^r_{n-1}(z).$$

Take a point $z_0 \in \mathbb{S}^{n-1}_{\sqrt{1-|y|^2}} \cap \partial\big(B_n(\frac{1}{\sqrt{2}}e_1, \frac{1}{2\sqrt{2}R})\big)$, and let θ be the angle between vectors z_0 and e_1, which the largest one between $z \in E$ and e_1. Here ∂B is the boundary of a set B. Then θ is small if R is large and[2] $|E| \sim (\sqrt{1-|y|^2}\theta)^{n-1} \sim \theta^{n-1}$. Noticing that $\theta^2 \sim \sin^2\theta = 1-\cos^2\theta \sim 1-\cos\theta$ and that

$$1-|y|^2 + \tfrac{1}{2} - \sqrt{2}\sqrt{1-|y|^2}\cos\theta = \tfrac{1}{8R^2},$$

we obtain that $\theta^2 \sim \frac{1}{8R^2} - (\sqrt{1-|y|^2} - \frac{1}{\sqrt{2}})^2$. Then we write

$$\Big|\sqrt{1-|y|^2} - \tfrac{1}{\sqrt{2}}\Big| = \Big|\frac{1-|y|^2-\frac{1}{2}}{\sqrt{1-|y|^2}+\frac{1}{\sqrt{2}}}\Big| \leq 2|\tfrac{1}{2} - |y|^2| \leq \tfrac{1}{25R}.$$

Consequently $\theta \geq C/R$.

[2] $A \sim B$ means that the ratio A/B is bounded above and below

Collecting the previous calculations, we can bound I from below by

$$\int_0^\theta \int_{\mathbb{S}^{n-2}_{t\sin\alpha}} |\sqrt{2}Rz - Re_1|^{-\frac{n}{p_2}} (-\log|Re_1 - \sqrt{2}Rz|)^{-\frac{2}{p_2}} d\sigma^{t\sin\alpha}_{n-2}(z) d\alpha,$$

where $t = |z| = \sqrt{1-|y|^2} \approx \frac{1}{\sqrt{2}}$, and $z_1 = \cos\alpha$. By symmetry, let us consider just that case $t < \frac{1}{\sqrt{2}}$. Let β be the angle such that $|\sqrt{2}z - e_1| = 2|\sqrt{2}t - 1|$, then $2t^2 + 1 - 2\sqrt{2}t\cos\beta = 4|\sqrt{2}t-1|^2$, which implies that $\beta^2 \sim 1 - \cos\beta \sim 2\sqrt{2}t - 2t^2 - 1 + 4(\sqrt{2}t-1)^2 = 3(\sqrt{2}t-1)^2$. So $\beta \sim 1 - \sqrt{2}t$. When $\alpha = 0$, we have trivially that $|\sqrt{2}z - e_1| = |\sqrt{2}t - 1|$. So for $\alpha \in [0,\beta]$, we have $|\sqrt{2}z - e_1| \sim 2|\sqrt{2}t - 1| \le 2\big|2|z|^2 - 1\big| = 2\big|2|y|^2 - 1\big| \le 6\big|\sqrt{2}|y| - 1\big| \le 6|\sqrt{2}y - e_1|$. Consequently using the fact that $1 - \sqrt{2}t \le C\theta$ and (10) again we obtain

$$\begin{aligned}
I &\ge C\int_0^\theta \int_{\mathbb{S}^{n-2}_{t\sin\alpha}} \frac{|\sqrt{2}Rz - Re_1|^{1-n}}{|\sqrt{2}Rz - Re_1|^{\frac{n}{p_2}-n+1}(-\log|Re_1 - \sqrt{2}Rz|)^{\frac{2}{p_2}}} \, d\sigma^{t\sin\alpha}_{n-2}(z)\, d\alpha \\
&\ge \frac{CR^{1-n}|\sqrt{2}t-1|^{1-n}}{|\sqrt{2}Ry - Re_1|^{\frac{n}{p_2}-n+1}(-\log|Re_1 - \sqrt{2}Ry|)^{\frac{2}{p_2}}} \int_0^{C(1-\sqrt{2}t)} \sin^{n-2}\alpha d\alpha \\
&\ge CR^{1-n} \frac{|\sqrt{2}t-1|^{1-n}|1-\sqrt{2}t|^{n-1}}{|\sqrt{2}Ry - Re_1|^{\frac{n}{p_2}-n+1}(-\log|Re_1 - \sqrt{2}Ry|)^{\frac{2}{p_2}}} \\
&= CR^{1-n}|\sqrt{2}Ry - Re_1|^{-\frac{n}{p_2}+n-1}(-\log|Re_1 - \sqrt{2}Ry|)^{-\frac{2}{p_2}}.
\end{aligned}$$

Using this estimate we see that

$$\begin{aligned}
&M_{\sqrt{2}R}(f,g)(Re_1) \\
&\quad\ge CR^{1-n}\int_{B_n(\frac{1}{\sqrt{2}}e_1, \frac{1}{100R})} |Re_1 - \sqrt{2}Ry|^{-\frac{n}{p}+n-1}(-\log|Re_1 - \sqrt{2}Ry|)^{-\frac{2}{p}} dy \\
&\quad= CR^{1-2n}\int_{B_n(0,\frac{1}{100})} |x|^{-\frac{n}{p}+n-1}(-\log|x|)^{-\frac{2}{p}} dx \\
&\quad= CR^{1-2n}\int_0^{\frac{1}{100}} r^{-\frac{n}{p}+2n-2}(-\log r)^{-\frac{2}{p}} dr \\
&\quad= \begin{cases} CR^{-2n+1} & \text{if } p = \frac{n}{2n-1} \\ \infty & \text{if } p < \frac{n}{2n-1} \end{cases}.
\end{aligned}$$

Hence $\mathcal{M}(f,g)$ is not in L^p for $p < \frac{n}{2n-1}$ and $\mathcal{M}(f,g)(x) \ge C|x|^{1-2n}$ for all $|x|$ large enough, hence it is also not in $L^{\frac{n}{2n-1}}(\mathbb{R}^n)$ when $p = \frac{n}{2n-1}$. □

Lastly, we prove a couple of points left open.

Lemma 8. *Let $r_1, r_2 > 0$, $t,\ s \le \frac{1}{10}$, and $t \le Cs$ for some $C \ge 1$. Then there exists an absolute constant C' (depending on C, r_1, r_2) such that*

$$s^{-r_1}(\log \tfrac{1}{s})^{-r_2} \le C' t^{-r_1}(\log \tfrac{1}{t})^{-r_2}. \tag{10}$$

Proof. Define $F(x) = x^{r_1}(\log x)^{-r_2}$. Differentiating F, we see that F is increasing when x is large enough and so,

$$F(\tfrac{1}{s}) = s^{-r_1}(\log \tfrac{1}{s})^{-r_2} \le C^{r_1}(Cs)^{-r_1}(\log \tfrac{1}{Cs})^{-r_2} = C^{r_1}F(\tfrac{1}{Cs}) \le C' F(\tfrac{1}{t}),$$

which is a restatement of (10). □

Lemma 9. *For functions $F(y,z)$ defined in $\mathbb{R}^{2n}$ with $y,\ z \in \mathbb{R}^n$, we have*

$$\int_{\mathbb{S}^{2n-1}} F(y,z)\,d\sigma(y,z) = \int_{B_n}\int_{\mathbb{S}^{n-1}_{r_y}} F(y,z)\,d\sigma^{r_y}_{n-1}(z)\frac{dy}{\sqrt{1-|y|^2}}, \tag{11}$$

where B_n is the unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1}_{r_y}$ is the sphere in $\mathbb{R}^n$ centered at 0 with radius $r_y = \sqrt{1-|y|^2}$.

Proof. We begin by writing $\int_{\mathbb{S}^{2n-1}} F(y,z)\,d\sigma(y,z)$ as

$$\int_{B_{2n-1}} \big[F(y,z',z_n) + F(y,z',-z_n)\big]\frac{dy\,dz'}{\sqrt{1-|y|^2-|z'|^2}}, \tag{12}$$

where $z = (z', z_n)$, and $z_n = \sqrt{1-|y|^2-|z'|^2}$; see [11, Appendix D.5].

Writing $z/r_y = \omega = (\omega', \omega_{n-1}) \in \mathbb{R}^{n-1}\times\mathbb{R}$, we express the right hand side of (11) as

$$\begin{aligned}
&\int_{B_n}\int_{\mathbb{S}^{n-1}_{r_y}} F(y,z)\,d\sigma^{r_y}_{n-1}(z)\frac{dy}{\sqrt{1-|y|^2}}\\
=&\int_{B_n} r_y^{n-1}\int_{\mathbb{S}^{n-1}} F(y,r_y\omega)\,d\sigma_{n-1}(\omega)\frac{dy}{\sqrt{1-|y|^2}}\\
=&\int_{B_n} r_y^{n-1}\int_{B_{n-1}} \big[F(y,r_y\omega',r_y\omega_n) + F(y,r_y\omega',-r_y\omega_n)\big]\frac{d\omega'}{\sqrt{1-|\omega'|^2}}\frac{dy}{\sqrt{1-|y|^2}}\\
=&\int_{B_n} r_y^{n-1}\int_{r_yB_{n-1}} \big[F(y,z',z_n) + F(y,z',-z_n)\big]\frac{r_y^{1-n}dz'}{\sqrt{1-|\omega'|^2}}\frac{dy}{\sqrt{1-|y|^2}}\\
=&\int_{B_n}\int_{r_yB_{n-1}} \big[F(y,z',z_n) + F(y,z',-z_n)\big]\frac{dy\,dz'}{\sqrt{1-|y|^2-|z'|^2}},
\end{aligned}$$

as one can easily verify that $\sqrt{1-|\omega'|^2}\sqrt{1-|y|^2} = \sqrt{1-|y|^2-|z'|^2}$. Using that B_{2n-1} is equal to the disjoint union of the sets $\{(y, r_y v):\ v \in B_{n-1}\}$ over all $y \in B_n$, we see that the last double integral is equal to the expression in (12), as claimed. □

The restriction $n \geq 8$ is due to the form of δ_n of Proposition 4, which relies on the exponent $1/5$ in Proposition B. An improvement of this exponent would help lower the dimension in Theorem 1.
Conjecture. The smallest δ_n in Proposition 4 is $n-1$. This would imply that $\mathcal{M}(f,g)$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$ when $n \geq 2$.

7. Appendix: Proof of Proposition B

Proposition B is contained in [13], whose proof is implicitly contained in [13, Section 4], but we outline it here only for the sake of completeness. The proof is based on wavelets with compact supports first constructed by Daubechies [7]. For our purposes, the wavelets need to be of product type and the exact form we use can be found in Triebel [25].

Lemma 10. *For any fixed $k \in \mathbb{N}$ there exist real compactly supported functions $\psi_F, \psi_M \in \mathcal{C}^k(\mathbb{R})$, which satisfy $\|\psi_F\|_{L^2(\mathbb{R})} = \|\psi_M\|_{L^2(\mathbb{R})} = 1$, for $0 \leq \alpha \leq k$ we have $\int_{\mathbb{R}} x^\alpha \psi_M(x)dx = 0$, and, if Ψ^G is defined by*

$$\Psi^G(\vec{x}) = \psi_{G_1}(x_1)\cdots\psi_{G_{2n}}(x_{2n})$$

for $G = (G_1, \dots, G_{2n})$ in the set

$$\mathcal{I} := \Big\{(G_1, \dots, G_{2n}) : \ G_i \in \{F, M\}\Big\},$$

then the family of functions

$$\bigcup_{\vec{\mu}\in\mathbb{Z}^{2n}} \left[\Big\{\Psi^{(F,\dots,F)}(\vec{x}-\vec{\mu})\Big\} \cup \bigcup_{\lambda=0}^{\infty} \Big\{2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu}) : \ G\in\mathcal{I}\setminus\{(F,\dots,F)\}\Big\}\right]$$

forms an orthonormal basis of $L^2(\mathbb{R}^{2n})$, where $\vec{x} = (x_1, \dots, x_{2n})$. We use also the notation $\Psi_{\vec{\mu}}^{\lambda,G} = 2^{\lambda n}\Psi^G(2^\lambda\vec{x}-\vec{\mu})$

Lemma 11. *Assume m is as in Proposition B. Then for any $j \in \mathbb{Z}$ and $\lambda \in \mathbb{N}_0$ we have*

$$|\langle\Psi_{\vec{\mu}}^{\lambda,G}, m\rangle| \leq CC_M 2^{-(M+1+n)\lambda}, \tag{13}$$

where $M = 4n$ is the number of vanishing moments of ψ_M.

We delete the simple verification of this lemma. The interested reader may refer to [13, Lemma 7] for details.

We are now ready to prove Proposition B.

The set $\mathcal{I}$ is finite, and the wavelets are compactly supported, so we may fix the type, namely G, of the wavelet and may assume futher that $m = \sum_{\lambda\geq 0}\sum_{D_\lambda} a_\omega \omega$ such that the supports of a_ω and $a_{\omega'}$ are disjoint when $\omega, \ \omega' \in D_\lambda$ and $\omega \neq \omega'$.

The level parameter is denoted by λ. Each ω at a fixed level λ is of tensor product type, i.e., $\omega = \omega_1\omega_2$. So, we can index ω_1 and ω_2 by $k,l \in \mathbb{Z}^n$, in such way that $\omega = \omega_{k,l} = \omega_{1,k}\omega_{2,l}$. Correspondingly we have $a = \{a_{(k,l)}\}_{k,l}$ with $a_{(k,l)} = \langle \omega_{1,k}\omega_{2,l}, m\rangle$. Moreover, we see that $\|a\|_2 = \|a\|_{\ell^2} \le \|m\|_2$ since $\{\omega\}$ is an orthonormal basis, and $\|a\|_\infty = \|a\|_{\ell^\infty} \le CC_M 2^{-(M+1+n)\lambda}$ by Lemma 11.

Now for $r \ge 0$ we define sets

$$U_r = \{(k,l) \in \mathbb{Z}^{2n} : 2^{-r-1}\|a\|_\infty < |a_{(k,l)}| \le 2^{-r}\|a\|_\infty\}.$$

From the ℓ^2 norm of a, we see $\#U_r \lesssim 2^{2r}\|a\|_2^2/\|a\|_\infty^2$. Let

$$N = 2^{r/4}\big(\tfrac{\|a\|_2}{\|a\|_\infty}\big)^{2/5}.$$

We split each $U_r = U_r^1 \cup U_r^2 \cup U_r^3$, where

$$U_r^1 = \{(k,l) \in U_r : \#\{s : (k,s) \in U_r\} \ge N\},$$

$$U_r^2 = \{(k,l) \in U_r \setminus U_r^1 : \#\{s : (s,l) \in U_r \setminus U_r^1\} \ge N\}.$$

and the third set U_r^3 is the remainder.

Let $E = \{k : (k,l) \in U_r^1\}$. Let $N_1 = \#E \le 2^{2r}\|a\|_2^2/(\|a\|_\infty^2 N)$. We now write $m_{r,1} = \sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{1,k}\omega_{2,l}$. Then

$$\begin{aligned}
\|T_{m_{r,1}}(f,g)\|_{L^1} \le & \Big\| \sum_{(k,l)\in U_r^1} a_{(k,l)}\mathcal{F}^{-1}(\omega_{1,k}\widehat{f})\mathcal{F}^{-1}(\omega_{2,l}\widehat{g})\Big\|_{L^1} \\
\le & \sum_{k\in E} \|\omega_{1,k}\widehat{f}\|_{L^2}\Big\| \sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{2,l}\widehat{g}\Big\|_{L^2} \\
\le & \Big(\sum_{k\in E} 1\Big)^{1/2}\Big(\sum_{k\in E}\big\|\omega_{1,k}\widehat{f}\big\|_{L^2}^2\Big)^{1/2} 2^{\lambda n/2}2^{-r}\|a\|_\infty\|g\|_{L^2} \\
\le & CN_1^{1/2}2^{-r}2^{\lambda n}\|a\|_\infty\|f\|_{L^2}\|g\|_{L^2} \\
\le & C2^{-r/8}2^{-M_1\lambda}\|a\|_2^{4/5}C_M^{1/5}\|f\|_{L^2}\|g\|_{L^2}.
\end{aligned}$$

Notice that here to estimate $\|\sum_{(k,l)\in U_r^1} a_{(k,l)}\omega_{2,l}\widehat{g}\|_{L^2}$ we use that for each fixed k, the supports of $\omega_{2,l}$ with $(k,l) \in U_r^1$ are disjoint and that $\|\omega_{2,l}\|_{L^\infty} \sim 2^{\lambda n}$.

The set U_r^2 is handled in the same way.

By the definition of U_r^3, for each (k,l) in it with k fixed there exist at most N pairs (k,l') in U_r^3, and with l fixed we have at most N pairs (k',l) in U_r^3. Then we can decompose $U_r^3 = \cup_{s=1}^{N^2} V_s$ such that if $(k,l),(k',l') \in V_s$ then $(k,l) \ne (k',l')$ implies $k \ne k'$ and $l \ne l'$. Associated to each V_s, there is a corresponding multiplier m_{V_s} and a bilinear operator $T_{m_{V_s}}$.

$$\begin{aligned}\left\|T_{m_{V_s}}(f,g)\right\|_{L^1} &\leq \sum_{(k,l)\in V_s} |a_{(k,l)}| \left\|\mathcal{F}^{-1}(\omega_{1,k}\widehat{f})\mathcal{F}^{-1}(\omega_{2,l}\widehat{g})\right\|_{L^1} \\ &\leq C2^{-r}\|a\|_\infty \Big[\sum_{(k,l)\in V_s} \left\|\omega_{1,k}\widehat{f}\right\|_{L^2}^2\Big]^{\frac{1}{2}} \Big[\sum_{(k,l)\in V_s} \left\|\omega_{2,l}\widehat{g}\right\|_{L^2}^2\Big]^{\frac{1}{2}} \\ &\leq C2^{-r}2^{\lambda n}\|a\|_\infty \|f\|_{L^2}\|g\|_{L^2}.\end{aligned}$$

Summing over s yields

$$\begin{aligned}\left\|T_{m_{r,3}}(f,g)\right\|_{L^1} &\leq N^2 2^{-r}2^{\lambda n}\|a\|_\infty \|f\|_{L^2}\|g\|_{L^2} \\ &\leq C2^{-r/2}2^{-M_1\lambda}\|a\|_2^{4/5}C_M^{1/5}\|f\|_{L^2}\|g\|_{L^2},\end{aligned}$$

which is also a good decay.

Summing over r and λ in order, we obtain (5).

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REFERENCES

[1] Bourgain, J., *Averages in the plane over convex curves and maximal operators*. Journal d'Analyse Mathématique **47**(1) (1986), 69–85.

[2] Calderón, C. P., *Lacunary spherical means*. Illinois Journal of Mathematics **23**(3) (1979), 476–484.

[3] Carbery, A., *Radial Fourier multipliers and associated maximal functions*. North-Holland Mathematics Studies **111** (1985), 49–56.

[4] Cho, Y. K., *A sharp (H^1, L^1) continuity theorem for the Rubio de Francia maximal multiplier operator*. J. Korean Math. Soc. **32** (3) (1995), 615–623.

[5] Coifman, R. R. and Weiss, G., Review: R. E. Edwards and G. Gaudry, *Littlewood-Paley and multiplier theory*. Bulletin of the American Mathematical Society **84**(2) (1978), 242–250.

[6] Cowling, M. and Mauceri, G., *On maximal functions*. Milan Journal of Mathematics **49**(1) (1979), 79–87.

[7] Daubechies, I., *Ten lectures on wavelets*, CBMS-NSF Regional Conference Series in Applied Mathematics, 61. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

[8] Duoandikoetxea, J. and Rubio de Francia, J.-L., *Maximal and singular integral operators via Fourier transform estimates*. Inventiones Mathematicae **84** (1986), 541–561.

[9] Duoandikoetxea, J. and Vega, L., *Spherical means and weighted inequalities*. Journal of the London Mathematical Society **53**(2) (1996), 343–353.

[10] Geba, D., Greenleaf, A., Iosevich, A., Palsson, E. and Sawyer, E. , *Restricted convolution inequalities, multilinear operators and applications*. Mathematical Research Letters **20** (2013), no. 4, 675–694.

[11] Grafakos, L., *Classical Fourier Analysis*, Third Edition. Graduate Texts in Mathematics, **249**, Springer, New York, 2014.

[12] Grafakos, L., *Modern Fourier analysis*, Third edition, Graduate Texts in Mathematics, **250**, Springer, New York, 2014.
[13] Grafakos, L., He, D., and Honzík P., *Rough bilinear singular integrals*. Submitted (2016).
[14] Grafakos, L., He, D., and Honzík P., *The Hörmander multiplier theorem, II: The bilinear local L^2 case*. Submitted (2016).
[15] Greenleaf, A., *Principal curvature and harmonic-analysis*. Indiana University Mathematics Journal **30**(4) (1981), 519–537.
[16] Greenleaf, A., Iosevich, A., Krause, B., and Liu, A., *Bilinear generalized Radon transforms in the plane*. https://arxiv.org/abs/1704.00861
[17] He, D., *On bilinear maximal Bochner-Riesz operators*. Submitted (2016).
[18] Magyar, A., Stein, E., and Wainger, S., *Discrete analogues in harmonic analysis: spherical averages*. Annals of Mathematics (2nd Ser.) **155**(1) (2002), 189–208.
[19] Mockenhaupt, G., Seeger, A., and Sogge, C. D., *Wave front sets, local smoothing and Bourgain's circular maximal theorem*. Annals of Mathematics (2nd Ser.) **136**(1) (1992), 207–218.
[20] Oberlin, D., *Multilinear convolutions defined by measures on spheres*. Transactions of the American Mathematical Society, 310 (1988), pp. 821–835.
[21] Rubio de Francia, J. L., *Maximal functions and Fourier transforms*. Duke Mathematical Journal **53**(2) (1986), 395–404.
[22] Schlag, W., *A geometric proof of the circular maximal theorem*. Duke Mathematical Journal **93**(3) (1998), 505–534.
[23] Stein, E. M., *Maximal functions: spherical means*. Proceedings of the National Academy of Sciences, **73**(7) (1976), 2174–2175.
[24] Stein, E. M., *Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton Mathematical Series, 43, Monographs in Harmonic Analysis, III, Princeton University Press, Princeton, NJ, 1993.
[25] Triebel, H., *Theory of function spaces. III, vol. 100 of Monographs in Mathematics*, 2006.

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